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# CONTINUUM MODELS OF POROUS MEDIA

F. Vodák

Czech Technical University (CVUT), Faculty of Civil Engineering, Prague, 166 29 Praha 6, Czechoslovakia

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**Abstract**—The continuum theory of porous materials respecting the internal discontinuity of these materials, is formulated in this paper. It is shown that porous material can be studied in two classes of continua (continua associated with volume and mass average quantities), five different kinds existing in each class. The relation of these classes and kinds to various parts and configurations of the porous medium is defined. Transformation equations between the classes and kinds are determined. The value of the principle of superposition of "components" is discussed and the principle of superposition of states is formulated.

## 1. INTRODUCTION

Porous materials are usually studied by aid of mixture theory (see, e.g. [1–3]†). This procedura leads to a whole series of difficulties and inaccuracies which were pointed out in [6, 7]. In these two studies it was shown that a continuum (or more accurately macrocontinuum) model can be co-ordinated to a porous material only if the following condition is satisfied in measuring the properties of the medium:

$$l \ll L \tag{1.1}$$

where $l$ is the internal characteristic length of the porous medium (e.g. the mean radius of the pores) and $L$ is the characteristic length of measuring probe (e.g. the wavelength of an acoustic signal). If we measure the property in accordance with condition (1.1), the measured values can then be associated with a continuum model of the material. Several such models, which are created if the volume average of the property is being measured, were discussed in [7]. However, from the results in [6] it follows that in balance equations not only volume averages, but also mass averages of intensive variables occur together. Therefore, in the subsequent section we shall seek to determine the possible classes and types of continua which are associated with volume and mass averages alike, we shall attempt to express their mutual relationship and we shall evaluate all obtained results from the point of view of their interpretability and measurability.

## 2. MATERIALS AND ITS MODELS

Part $\mathcal{M}$ of a porous material at time $t$ occupies a region $\mathfrak{R}$ which consists of volume $\Delta V$ and its surface $\delta\Delta V$. Region $\mathfrak{R}$ consists of a continuous region $\mathfrak{R}_1$ (with the volume $\Delta V_1$) which corresponds to a solid skeleton, and of $N$ discrete regions $\mathfrak{R}_2{}^i$ (with volumes $\Delta V_2{}^i$) corresponding to the pore space. The fluid in the pore space is in two different phases, liquid and gas. Let us, therefore, designate the individual $\mathfrak{R}_2{}^i$ that the first $X$ corresponds to the liquid and the remaining ones to the gas. It holds that

$$\begin{gathered}\mathfrak{R} = \sum_{\alpha=1}^{2} \mathfrak{R}_\alpha,\ \mathfrak{R}_2 = \sum_{i=1}^{N} \mathfrak{R}_2{}^i,\ \mathfrak{R}_2{}^I = \sum_{i=1}^{X} \mathfrak{R}_2{}^i,\\ \mathfrak{R}_2{}^{II} = \sum_{i=X+1}^{N} \mathfrak{R}_2{}^i,\ \Delta V_2{}^I = \sum_{i=1}^{X} \Delta V_2{}^i,\ \Delta V_2{}^{II} = \sum_{i=X+1}^{N} \Delta V_2{}^i,\\ \Delta V = \sum_{\alpha=1}^{2} \Delta V_\alpha,\ \Delta V_2 = \sum_{i=1}^{N} \Delta V_2{}^i = \sum_{\omega=1}^{II} \Delta V_2{}^\omega.\end{gathered} \tag{2.1}$$

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†Biot's pioneering work [4] is likewise based on a mixture model (see [5]).

The total mass of $\mathcal{M}$ is $\Delta M$ and the mass contained in $\mathcal{R}_\alpha$ or $\mathcal{R}_\alpha{}^i$ or $\mathcal{R}_2{}^\omega$ is $\Delta M_\alpha$ or $\Delta M_\alpha{}^i$ or $\Delta M_2^{\omega\alpha}$, respectively and it holds that

$$\Delta M = \sum_{\alpha=1}^{2} \Delta M_\alpha, \; \Delta M_2 = \sum_{i=1}^{N} \Delta M_2{}^i = \sum_{\omega\alpha=1}^{II} \Delta M_2{}^\omega,$$

$$\Delta M_2{}^I = \sum_{i=1}^{X} \Delta M_2{}^i, \; M_2{}^{II} = \sum_{i=X+1}^{N} \Delta M_2{}^i. \tag{2.2}$$

From now on we shall distinguish between measurements at the macropoint of the material (i.e. if condition (1.1) is satisfied) and at the micropoint (i.e. if condition (1.1) is not satisfied). We shall denote an intensive variable measured at a micropoint of $\mathcal{R}$ or $\mathcal{R}_1$ or $\mathcal{R}_2{}^i$ as $A = A(\mathbf{r}, t)$ for $\mathbf{r} \in \mathcal{R}\forall$) or $A_1 = A_1(\mathbf{r}, t)$ (for $\mathbf{r} \in \mathcal{R}_1$), or $A_2{}^i = A_2{}^i(\mathbf{r}, t)$ (for $\mathbf{r} \in \mathcal{R}_2{}^i$). It holds that $A = A_1 \vee A_2$, where $A_2 = A_2{}^1 \vee \ldots \vee A_2{}^N$. If we define the unit functions as

$$D_\alpha(\mathbf{r}, t) \equiv \begin{cases} 1 \text{ for } \mathbf{r} \in \mathcal{R}_\alpha \\ 0 \text{ for } \mathbf{r} \notin \mathcal{R}_\alpha \end{cases}, \; D_2{}^i(\mathbf{r}, t) \equiv \begin{cases} 1 \text{ for } \mathbf{r} \in \mathcal{R}_2{}^i \\ 0 \text{ for } \mathbf{r} \neq \mathcal{R}_2{}^i \end{cases} \tag{2.3}$$

for an arbitrary $\mathbf{r}$ it then holds that

$$\sum_\alpha D_\alpha = 1, \; \sum_i D_\alpha{}^i = D_\alpha, \; A = \sum_\alpha A_\alpha^*, \; A_\alpha^* \equiv D_\alpha A_\alpha$$

$$A_2^* = \sum_i A_2{}^{i*}, \; A_2{}^{i*} \equiv D_2{}^i A_2{}^i. \tag{2.4}$$

If we are carrying out the measurements in accordance with condition (1.1), depending on the nature of these measurements we may then define various types of continua which will be associated with the following volume and mass averages of the variables:†

$$\langle A \rangle \equiv \lim \frac{1}{\Delta V} \int_{\Delta V} A \, \mathrm{d}V \tag{2.5}$$

$$\langle A_\alpha^* \rangle \equiv \lim \frac{1}{\Delta V} \int_{\Delta V} A_\alpha^* \, \mathrm{d}V \tag{2.6}$$

$$\langle A_2{}^{i*} \rangle \equiv \lim \frac{1}{\Delta V} \int_{\Delta V} A_2{}^{i*} \, \mathrm{d}V \tag{2.7}$$

$$\langle A_\alpha \rangle_\alpha \equiv \lim \frac{1}{\Delta V_\alpha} \int_{\Delta V_\alpha} A_\alpha \, \mathrm{d}V \tag{2.8}$$

$$\langle A_2{}^i \rangle_2{}^i \equiv \lim \frac{1}{\Delta V_2{}^i} \int_{\Delta V_2{}^i} A_2{}^i \, \mathrm{d}V \tag{2.9}$$

$$\left\{ A \right\} \equiv \lim \frac{1}{\Delta M} \int_{\Delta M} A \, \mathrm{d}M \tag{2.10}$$

$$\left\{ A_\alpha^* \right\} \equiv \lim \frac{1}{\Delta M} \int_{\Delta M} A_\alpha^* \, \mathrm{d}M \tag{2.11}$$

$$\left\{ A_2{}^{i*} \right\} \equiv \lim \frac{1}{\Delta M} \int_{\Delta M} A_2{}^{i*} \, \mathrm{d}M \tag{2.12}$$

$$\left\{ A_\alpha \right\}_\alpha \equiv \lim \frac{1}{\Delta M_\alpha} \int_{\Delta M_\alpha} A_\alpha \, \mathrm{d}M \tag{2.13}$$

$$\left\{ A_2{}^i \right\}_2^i \equiv \lim \frac{1}{\Delta M_2{}^i} \int_{\Delta M_2{}^i} A_2{}^i \, \mathrm{d}M. \tag{2.14}$$

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†Unless otherwise indicated the limits are calculated for $l/L \to 0$.

The continua $\mathscr{K}_{\langle\rangle}$ and $\mathscr{K}_{\{\}}$ are associated with the quantities defined by relations (2.5) and (2.10). The continua $\mathscr{K}^*_{\alpha\langle\rangle}$, and $\mathscr{K}^*_{\alpha\{\}}$, $\mathscr{K}^{i*}_{2\langle\rangle}$, and $\mathscr{K}^{i*}_{2\{\}}$, $\mathscr{K}_{\alpha\langle\rangle}$ and $\mathscr{K}_{\alpha\{\}}\mathscr{K}^i_{2\langle\rangle}$ and $\mathscr{K}^i_{2\{\}}$ correspond to the quantities (2.6) and (2.11), (2.7) and (2.12), (2.8) and (2.13), and (.29) and (2.14), respectively.

If we define the following volume and mass concentrations

$$\delta_\alpha \equiv \langle D_\alpha \rangle = \lim \frac{\Delta V_\alpha}{\Delta V},\ \delta_2{}^i \equiv \langle D_2{}^i \rangle = \lim \frac{\Delta V_2{}^i}{\Delta V} \tag{2.15}$$

$$c_\alpha \equiv \{D_\alpha\} = \lim \frac{\Delta M_\alpha}{\Delta M},\ c_2{}^i = \{D_2{}^i\} = \lim \frac{\Delta M_2{}^i}{\Delta M} \tag{2.16}$$

then from the relations (2.1)–(2.16) it follows that

$$\begin{aligned}\langle A \rangle = \sum_\alpha \langle A^*_\alpha \rangle,\ \langle A^*_\alpha \rangle = \delta_\alpha \langle A_\alpha \rangle_\alpha,\ \langle A^*_2 \rangle = \sum_i \langle A_2{}^{i*} \rangle = \delta_2 \langle A_2 \rangle_2 \\ = \sum_i \delta_2{}^i \langle A_2{}^i \rangle_2{}^i,\ \langle A_2{}^{i*} \rangle = \delta_2{}^i \langle A_2{}^i \rangle_2{}^i\end{aligned} \tag{2.17}$$

$$\begin{aligned}\{A\} = \sum_\alpha \{A^*_\alpha\},\ \{A^*_\alpha\} = c_\alpha \{A_\alpha\}_\alpha,\ \{A^*_2\} = \sum_i \{A_2{}^{i*}\} = c_2 \{A_2\}_2 \\ = \sum_i c_2{}^i \{A_2{}^i\},\ \{A_2{}^{i*}\} = c_2{}^i \{A_2{}^i\}_2{}^i,\end{aligned} \tag{2.18}$$

$$\sum_\alpha \delta_\alpha = 1,\ \sum_i \delta_2{}^i = \delta_2,\ \sum_\alpha c_\alpha = 1,\ \sum_i c_2{}^i = c_2. \tag{2.19}$$

For the time being we have defined the two main classes of continua: the continua corresponding to measurements in characteristic volumes and masses. Within each class we have distinguished five different types of continua and eqns (2.17) and (2.18) determine the relations between the individual types of a single class. However, it still is not clear which of the individual measurements can be realized or, in other words, which continua can be interpreted in terms of measurable quantities. This problem will be discussed in the next section.

## 3. INTERPRETATION OF CONTINUUM MODELS

Figure 1 illustrates the porous material $\mathscr{M}$. $\mathscr{M}$ can be divided (at least in one's mind) into materials $\mathscr{M}_1$ and $\mathscr{M}_2$, which correspond to a solid skeleton and a fluid in the pore space. $\mathscr{M}_2$ can further be divided into $N$ materials $\mathscr{M}_2{}^i$, each of which corresponds to a certain pore or its part (i.e. the regions $\mathscr{R}_2{}^i$). From $\mathscr{M}_\alpha$ we can now proceed to material $\mathscr{M}^*_\alpha$. $\mathscr{M}^*_1$ is material $\mathscr{M}_1$ in region $\mathscr{R}$ (i.e. there is a vacuum in $\mathscr{R}_2$) and similarly $\mathscr{M}^*_2$ is $\mathscr{M}_2$ in region $\mathscr{R}$ (i.e. there is a vacuum in $\mathscr{R}_1$), If we are considering $\mathscr{M}_2{}^i$ in $\mathscr{R}$, we then obtain material $\mathscr{M}_2{}^{i*}$, i.e. a material in which there is a vacuum everywhere within the boundaries of $\delta\mathscr{M}\Delta V$ except for the region $\mathscr{R}_2{}^i$. If we are measuring at the micropoints of material $\mathscr{M}_\alpha$, $\mathscr{M}_2{}^i$, $\mathscr{M}^*_\alpha$, and $\mathscr{M}_2{}^{i*}$, we then obtain the quantities $A_\alpha$, $A_2{}^i$, $A^*_\alpha$ and $A_2{}^{i*}$, respectively. If we are measuring at the macropoints of $\mathscr{M}$, we then obtain either $\langle A \rangle$ or $\{A\}$, i.e the continua $\mathscr{K}_{\langle\rangle}$ and $\mathscr{K}_{\{\}}$ are associated with $\mathscr{M}$. The results of the measurements at the macropoints of $\mathscr{M}_\alpha$ are the values $\langle A_\alpha \rangle_\alpha$ and $\{A_\alpha\}_\alpha$, i.e. the continua $\mathscr{K}_{\alpha\langle\rangle}$ and $\mathscr{K}_{\alpha\{\}}$ are associated with $\mathscr{M}_\alpha$. The measurements at macropoint $\mathscr{M}_2{}^i$ lead to $\langle A_2{}^i \rangle_2{}^i$ or $\{A_2{}^i\}_2{}^i$ and, therefore, to the continua $\mathscr{K}^i_{2\langle\rangle}$ or $\mathscr{K}^i_{2\{\}}$. Measurements at the macropoints of materials $\mathscr{M}^*_\alpha$ and $\mathscr{M}_2{}^{i*}$ lead to the values $\langle A^*_\alpha \rangle$ and $\{A^*_\alpha\}$, $\langle A_2{}^{i*} \rangle$ and $\{A_2{}^{i*}\}$, i.e. to continua $\mathscr{K}^*_{\alpha\langle\rangle}$ and $\mathscr{K}^*_{\alpha\{\}}$, $\mathscr{K}^{i*}_{2\langle\rangle}$, $\mathscr{K}^{i*}_{2\{\}}$ and $\mathscr{K}^{i*}_{2\{\}}$.

In principle, the measurements of the quantities $\langle A \rangle$ and $\{A\}$ and of the concentrations $\delta_\alpha$, $\delta_2{}^i$, $c_\alpha$ and $c_2{}^i$ can be realized under all circumstances. Under certain circumstances, in the case of porous materials, we may measure the quantities $\langle A_1 \rangle_1$ and $\{A_1\}_1$, or $\langle A^*_1 \rangle$, or $\langle A_2{}^i \rangle_2{}^i$ and $\{A_2{}^i\}_2{}^i$, i.e we are able to measure the properties of the solid phase of a porous medium, or of the absolutely dry solid skeleton, or of the fluid in an individual pore (or in its part). By these

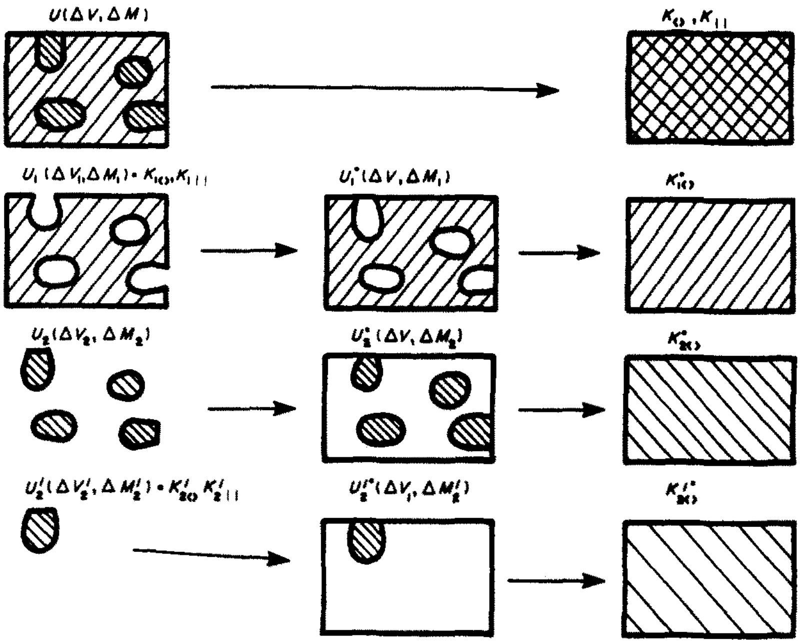

Fig. 1.

circumstances we understand certain conditions which must be satisfied during the experiment (e.g. in measuring $\langle A_1\rangle_1$ the boundary conditions of region $\mathfrak{R}_1$ in $\mathfrak{R}$ must be satisfied, etc.). The materials $\mathscr{M}_2$, $\mathscr{M}_2^*$ and $\mathscr{M}_2^{i*}$ are quite fictitious and unrealizable. It is impossible, for instance, to distribute the $\mathfrak{R}_2^i$'s, filled with fluid, in the vacuum region $\mathfrak{R}$ and at the same time preserve the boundary conditions, interaction forces, etc. which existed in $\mathscr{M}$. Neither is it possible to measure the quantities $\{A_1^*\}$ in material $\mathscr{M}_1^*$, because the latter has mass $\Delta M_1$, whereas $\{A_1^*\}$ is associated with mass $\Delta M$. To summarize, we may say that only the continua $\mathscr{K}_{\langle\rangle}$ (with the quantities $\langle A\rangle$) and $\mathscr{K}_{\{\}}$ ($\{A\}$) can be interpreted directly in terms of measurable quantities. With certain limitations (see above) we may also claim this in case of continua $\mathscr{K}_{1\langle\rangle}$ ($\langle A_1\rangle_1$), $\mathscr{K}_{1\{\}}$ ($\{A_1\}_1$), $\mathscr{K}_{1\langle\rangle}^*$ ($\langle A_1^*\rangle$), $\mathscr{K}_{2\langle\rangle}^i$ ($\langle A_2^i\rangle_2^i$) and $\mathscr{K}_{2\{\}}^i$ ($\{A_2^i\}_2^i$). The remaining continua (and the corresponding quantities) have only an auxiliary, or more accurately, a derived character (see eqns (2.17) and (2.18)).†

If we consider conditions $(2.1)_{3,4,6,7,8}$ and $(2.2)_{2,3,4}$, we may then define the properties $\langle A_2^I\rangle_2^I$, $\{A_2^I\}_2^I$ and the concentrations $\delta_2^I$ and $c_2^I$ of the liquid, as well as the properties $\langle A_2^{II}\rangle_2^{II}$, $\{A_2^{II}\}_2^{II}$ and concentrations $\delta_2^{II}$ and $c_2^{II}$ of the gas:

$$\sum_{i=1}^{X} \delta_2^i \langle A_2^i\rangle_2^i = \delta_2^I \langle A_2^I\rangle_2^I, \quad \sum_{i=X+1}^{N} \delta_2^i \langle A_2^i\rangle_2^i = c_2^{II} \langle A_2^{II}\rangle_2^{II}, \tag{3.1}$$

$$\sum_{i=1}^{X} c_2^i \{A_2^i\}_2^i = c_2^I \{A_2^I\}_2^I, \quad \sum_{i=X+1}^{N} c_2^i \{A_2^i\}_{2w}^i = c_2^{II} \{A_2^{II}\}_2^{II},$$

$$\delta_2^\omega = \lim \frac{\Delta V_2^\omega}{\Delta V}, \; c_2^\omega = \lim \frac{\Delta M_2^\omega}{\Delta M} \; (\omega = I, II),$$

$$\sum_{\omega=1}^{II} \delta_2^\omega = c_2, \; \sum_{\omega=I}^{II} c_2^\omega = c_2.$$

## 4. RELATIONS BETWEEN THE VOLUME AND MASS AVERAGE VARIABLES

First of all, we shall define the densities measured at the micropoints of the materials $\mathscr{M}$, $\mathscr{M}_\alpha$,

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†The relation of the individual continua to the concepts used in the theory of mixture will be discussed in the last section.

$\mathcal{M}_\alpha^*$, $\mathcal{M}_2{}^i$ and $\mathcal{M}_2{}^{i*}$:

$$\rho \equiv \lim_{\Delta V\to 0} \frac{\Delta M}{\Delta V},\ \rho_\alpha \equiv \lim_{\Delta V_\alpha \to 0} \frac{\Delta M_\alpha}{\Delta V_\alpha},$$

$$\rho_\alpha^* \equiv \lim_{\Delta V\to 0} \frac{\Delta M_\alpha}{\Delta V}, \rho_2{}^i \equiv \lim_{\Delta V_2{}^i \to 0} \frac{\Delta M_2{}^i}{\Delta V_2{}^i},\ \rho_2{}^{i*} \equiv \lim_{\Delta V\to 0} \frac{\Delta M_2{}^i}{\Delta V}. \tag{4.1}$$

The densities at the macropoints then read

$$\langle\rho\rangle = \lim \frac{\Delta M_\alpha}{\Delta V}, \langle\rho_\alpha\rangle_\alpha = \lim \frac{\Delta M_\alpha}{\Delta V_\alpha},\ \langle\rho_\alpha^*\rangle = \lim \frac{\Delta M_\alpha}{\Delta V}, \tag{4.2}$$

$$\langle\rho_2{}^i\rangle_2{}^i = \lim \frac{\Delta M_2{}^i}{\Delta V_2{}^i},\ \langle\rho_2{}^{i*}\rangle = \lim \frac{\Delta M_2{}^i}{\Delta V}.$$

Since we assumed that measurements of the elements of mass and volume could be realized, we may claim all the densities in eqns (4.2) to be measurable. It holds that (see (2.15), (2.16), (2.18), (4.2))

$$\begin{gathered} c_\alpha = \langle\rho_\alpha^*\rangle/\langle\rho\rangle,\ \delta_\alpha = \langle\rho_\alpha^*\rangle/\langle\rho_\alpha\rangle_\alpha, \\ c_2{}^i = \langle\rho_2{}^{i*}\rangle/\langle\rho\rangle,\ \delta_2{}^i = \langle\rho_2{}^{i*}\rangle/\langle\rho_2{}^i\rangle_2{}^i, \\ \delta_\alpha\langle\rho_\alpha\rangle_\alpha = c_\alpha\langle\rho\rangle,\ \delta_2{}^i = \langle\rho_2{}^i\rangle_2{}^i = c_2{}^i\langle\rho\rangle. \end{gathered} \tag{4.3}$$

Equations (2.5)–(2.19), (4.2) and (4.3) now yield

$$\begin{aligned} \langle\rho A\rangle = \langle\rho\rangle\{A\} &= \sum_\alpha \langle\rho_\alpha^* A_\alpha\rangle = \langle\rho_1^* A_1\rangle + \sum_i \langle\rho_2{}^{i*} A_2{}^i\rangle \\ &= \sum_\alpha \delta_\alpha\langle\rho_\alpha A_\alpha\rangle = \delta_1\langle\rho_1 A_1\rangle + \sum_i \delta_2{}^i\langle\rho_2{}^i A_2{}^i\rangle_2{}^i, \end{aligned} \tag{4.4}$$

$$\langle\rho_\alpha^* A_\alpha\rangle = \langle\rho_\alpha^*\rangle\{A_\alpha\}_\alpha = \langle\rho\rangle\{A_\alpha^*\} \tag{4.5}$$

$$\langle\rho_\alpha A_\alpha\rangle = \langle\rho_\alpha\rangle_\alpha\{A_\alpha\}_\alpha = \langle\rho_\alpha\rangle_\alpha c_\alpha{}^{-1}\{A_\alpha^*\} = \langle\rho\rangle\delta_\alpha{}^{-1}\{A_\alpha^*\} \tag{4.6}$$

$$\langle\rho_2{}^{i*} A_2{}^i\rangle = \langle\rho\rangle\{A_2{}^{i*}\} = \langle\rho_2{}^{i*}\rangle\{A_2{}^i\}_2{}^i \tag{4.7}$$

$$\langle\rho_2{}^i A_2{}^i\}_2 = \langle\rho_2{}^i\rangle_2{}^i\{A_2{}^i\}_2{}^i = \langle\rho\rangle(\delta_2{}^i)^{-1}\{A_2{}^{i*}\} = \langle\rho_2{}^i\rangle_2{}^i(c_2{}^i)^{-1}\{A_2{}^{i*}\}. \tag{4.8}$$

Provided we are able to state the hypothesis of microlocal homogeneity for the solid as well as the fluid, i.e. provided

$$\rho_1(\mathbf{r}) = \text{const.},\ \rho_2{}^i(\mathbf{r}) = \rho_2{}^j(\mathbf{r}) = \text{const.}, \tag{4.9}$$

it follows from eqns (2.8) and $(4.6)_1$ that

$$\langle A_\alpha\rangle_\alpha = \{A_\alpha\}_\alpha. \tag{4.10}$$

Equations $(2.17)_2$, $(2.18)_2$, $(2.19)_{3,4}$ and (4.10) then yield

$$\{A\} - \langle A\rangle = (c_1 - \delta_1)(\langle A_1\rangle_1 - \langle A_2\rangle_2). \tag{4.11}$$

Here it should be pointed out that conditions (4.9) can only be applied to a dry and liquid saturated porous medium.

## 5. REFERENCE STATES

The theory of mixture, built on the principle of superposition of components tends towards one very old dream of alchymists: it is attempting to express the property of the whole as a function of the properties of its parts. Relations (2.17), (2.18) and the analysis in Section 3 indicate that this is, in principle, possible even with porous materials, provided we are able to measure the quantities $\langle A_1\rangle_1$ and $\langle A_2{}^i\rangle_2{}^i$ (or $\{A_1\}_1$ and $\{A_2{}^i\}_2{}^i$). However, we did point out that it is quite difficult to make these measurements. For example, the water in pores is not identical with "free" water. For this reason the author is of the opinion that it is more convenient to replace the superposition of components by the superposition of reference states. As regards porous media, we are directly offered two realizable (and more or less reproducable[7]) states: a dry and liquid-saturated porous material.

For the quantities $\langle A\rangle$ playing the role of primitive terms in the given theory (independent variables), or the role of directly measurable quantities in an experiment, it holds that[7]

$$\langle A\rangle = \delta_P{}^I\langle A\rangle_S + \delta_P{}^{II}\langle A\rangle_D \tag{5.1}$$

where the indices $S$ and $D$ indicate quantities measured in the first and second reference state (i.e. in the liquid-saturated and dry porous material), $\delta_P{}^I$ and $\delta_P{}^{II}$ represent the liquid and gas contents of the pore space:

$$\delta_P{}^\omega = \delta_2{}^{-1}\delta_2{}^\omega = \lim \frac{\Delta V_2{}^\omega}{\Delta V_2}\,(\omega = I, II),\ \sum_{\omega=I}^{II} \delta_P{}^\omega = 1. \tag{5.2}$$

Rule (5.1) was derived under a single limiting assumption that the material does not swell. We shall also apply this assumption here in deriving the relation $\{A\} = f(\{A\}_S, \{A\}_D)$. Besides this, we shall assume that the liquid and the gas are microlocally homogeneous, i.e. that

$$\begin{aligned}
\langle\rho_2{}^i\rangle_2{}^i &= \langle\rho_2{}^I\rangle_2{}^I = \lim \frac{\Delta M_2{}^I}{\Delta V_2{}^I} = \lim \frac{\Delta M_2{}^{IS}}{\Delta V_2}\ (\text{for } i = 1, \ldots, X)\\
\langle\rho_2{}^i\rangle_2{}^i &= \langle\rho_2{}^{II}\rangle_2{}^{II} = \forall = \lim \frac{\Delta M_2{}^{II\forall D}}{\Delta V_2{}^{II}}\ (\text{for } i = X+1, \ldots, N).
\end{aligned} \tag{5.3}$$

Here $\Delta M_2{}^{IS}$ and $\Delta M_2{}^{IID}$ represent the mass of the liquid in the liquid-saturated material and of the gas in the dry material, respectively.

Equations (5.2) and (5.3) yield,

$$\delta_P{}^I = \lim \frac{\Delta V_2{}^I}{\Delta V_2} = \lim \frac{\Delta M_2{}^I}{\Delta M_2{}^{IS}},\ \delta_P{}^{II} = \lim \frac{\Delta V_2{}^{II}}{\Delta V_2} = \lim \frac{\Delta M_2{}^{II}}{\Delta M_2{}^{IID}}. \tag{5.4}$$

In agreement with eqns (2.18) and (3.1) we may write for the actual state of the material and for both reference states

$$\begin{aligned}
\{A\} &= c_1\{A_1\}_1 + c_2{}^I\{A_2{}^I\}_2{}^I + c^{2II}\{A_2{}^{II}\}_2{}^{II},\\
\{A\}_D &= c_1{}^D\{A_1\}_1 + c_2{}^{IID}\{A_2{}^{II}\}_2{}^{II},\\
\{A\}_S &= c_1{}^S\{A_1\}_1 + c_2{}^{IS}\{A_2{}^I\}_2{}^I,
\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}
c_1{}^D &= \lim \frac{\Delta M_1}{\Delta M^D},\ c_1{}^S = \lim \frac{\Delta M_1}{\Delta M^S},\\
c_2{}^{IID} &= \lim \frac{\Delta M_2{}^{IID}}{\Delta M^D},\ c_2{}^{IS} = \lim \frac{\Delta M_2{}^{IS}}{\Delta M^S}
\end{aligned} \tag{5.6}$$

where the masses of the liquid-saturated and dry media are

$$\Delta M^S = \Delta M_1 + \Delta M_2{}^{IS} \text{ and } \Delta M^D = \Delta M_1 + \Delta M_2{}^{IID}. \tag{5.7}$$

respectively. We may now say that (see eqns (5.2)–(5.7)) property $A$ of an arbitrary actual state is

$$\{A\} = \delta_P{}^I \frac{\langle\rho\rangle^S}{\langle\rho\rangle}\{A\}_S + \delta_P{}^{II}\frac{\langle\rho\rangle_D}{\langle\rho\rangle}\{A\}_D. \tag{5.8}$$

In [7] the dependence of some drived (indirectly measureable) quantities $\langle A\rangle$ on the values in reference states was studied (particular attention being paid to the strain tensor and material constants). It follows from eqns (5.1) and (5.8) that all results valid for $\mathcal{K}_0$ (see [7]) are as well valid for $\mathcal{K}_0$ if we make the transformation$\langle A\rangle \to \{A\}$, $\langle A\rangle_S \to \{A\}_S$, $\langle A\rangle_D \to \{A\}_D$,

$$\delta_P{}^I \to \delta_P{}^I \frac{\langle\rho\rangle_S}{\langle\rho\rangle},\ \delta_P{}^{II} \to \delta_P{}^{II}\frac{\langle\rho\rangle_D}{\langle\rho\rangle}.$$

## 6. DISCUSSION

The presented theory tried to respect the following facts:

(1) Measurements cannot be carried out in dimensionless points. Any measurement is associated with a finite volume or a finite mass.

(2) The validity of the principle of superposition of "components" is limited by the extent to which the "components" can be separated.

The first fact led us to two classes of continua (continua associated with volume and mass average quantities), each class being divided into five kinds. The relations between the kinds are expressed by eqns (2.17) and (2.18). Under certain conditions the relation between the classes (see (4.9) and (4.10)) has the form of (4.11). In the case of $A = \mathbf{v}$ ($\mathbf{v}$ is the velocity) this relation is identical with the equation derived by Brenner[8]. Equations (4.4)–(4.8) can be understood as relations between the densities of extensive variables and the specific values of these variables. The description of heterogeneous materials in terms of averaged quantities is not a new idea. It has been used, e.g. in papers[9] (for the description of general two-phase medium), [8, 10] (for porous media), [11, 12] (for immiscible mixtures of fluids). However, I think that the present article differs from the set of papers quoted above in one essential aspect—in the choice of primitive terms of theory. In the papers[8–12] the role of primitive terms is played by the variables measured in dimensionless points—so-called "exact" fields (e.g. the averaged quantities are expressed in terms of the "exact" fields, etc.). This choice was refused by the present paper (as it follows from all the arguments presented above). This methodological difference has determined differences between the applied mathematical instruments. For example, Drew[9] obtains continuously differentiable averaged functions from the "exact" variables by integrating over two volumes (see Reymond-Dubois Lemma). Here, it is supposed that averaged quantities are continuously differentiable as soon as the condition (1.1) is satisfied.† The difference mentioned above made the "program" of the present theory different from analogous theories, too. I should like to determine the content of concepts and their applicability limits before the derivation of balance equations. For example, the mass (or volume) average velocity is employed in papers[8–12]. But we measure rather "the velocity of averaging volume" (see [6, 7]).

The second fact forced us to replace the principle of superposition of components by the principle of superposition of reference states (see (5.1) and (5.8)). The results of this paper can also be understood to be a contribution to the argument conducted about the theory of mixtures between the non-equilibrium thermodynamics and the rational thermomechanics. The traditional (micro-contunuum) mixture theory utilizes the concept of the so-called partial continuum. In our macro-continuum formulations $\mathcal{K}^*_{\alpha 0}$ and $\mathcal{K}^*_{\alpha 0}$ correspond to these continua. Non-equilibrium thermodynamics uses the principle of superposition (and, therefore, of the partial continua) for balancing mass only. Of the partial quantities it thus uses only densities (or concentrations) and diffusion velocities. As our analysis has been shown these quantities can be measured in principle (the diffusion velocities by means of eqn (4.11)).‡ Rational ther-

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†The variables $A$, $A_\alpha$, etc. in eqns (2.5)–(2.14) are measured in micropoints, i.e. in some averaging volumes from point of view of "dimensionless" points.

‡The chemical potentials of the "components" also enter the entropy balance. Chemists claim that they are able to measure these as well.

momechanics employ partial continua in balancing all quantities. They therefore use quantities such as partial stress, partial heat flux, etc. Ooru analysis has shown that these quantities cannot be interpreted, because in most cases they cannot be measured. This conclusion is formulated only for porous materials, but the author is of the opinion that it is valid generally.

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